

NON-NEGATIVE GLOBAL WEAK SOLUTIONS FOR A DEGENERATE PARABOLIC SYSTEM MODELING THIN FILMS DRIVEN BY CAPILLARITY

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ABSTRACT. We prove global existence of non-negative weak solutions for a strongly coupled, fourth order degenerate parabolic system governing the motion of two thin fluid layers in a porous medium when capillarity is the sole driving mechanism.

1. INTRODUCTION AND THE MAIN RESULT

In this paper we study the following one-dimensional degenerate system of equations

$$\begin{cases} \partial_t f = -\partial_x [f \partial_x^3 (Af + Bg)], \\ \partial_t g = -\partial_x [g \partial_x^3 (f + g)], \end{cases} \quad (t, x) \in (0, \infty) \times (0, L), \quad (1.1)$$

which models the dynamics of two thin fluid threads in a porous medium in the absence of gravity. One of the fluids is located in the region bounded from below by the line $y = 0$ and from above by the graph $y = f(t, x)$, while the region occupied by the second fluid is located between the graphs $y = f(t, x)$ and $y = (f + g)(t, x)$, f and g being non-negative functions. Furthermore, L is a positive real number and the positive constants A and B have the following physical meaning

$$A := \frac{\mu_+}{\mu_-} \frac{\gamma_d + \gamma_w}{\gamma_d} > B := \frac{\mu_+}{\mu_-}.$$

We let μ_- [resp. μ_+] denote the viscosity of the fluid located below [resp. above], γ_w is the surface tension coefficient at the interface $y = f(t, x)$ between the wetting phases, while γ_d is the surface tension coefficient at the interface $y = (f + g)(t, x)$. The system (2.1) is supplemented by initial conditions

$$f(0) = f_0, \quad g(0) = g_0, \quad x \in (0, L), \quad (1.2)$$

whereby f_0 and g_0 are assumed to be known, and we impose no-flux boundary conditions

$$\partial_x f = \partial_x g = \partial_x^3 f = \partial_x^3 g = 0, \quad x = 0, L. \quad (1.3)$$

The system (1.1) has been obtained in [7], by passing to the small layer thickness in the Muskat problem studied in [6]. This is a widely used approach in the study of thin fluid threads because it reduces complex moving boundary value problems

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to local problems defined by generally simpler equations. System (1.1) is strongly related to the Thin Film equation because if, for instance, f is constantly equal to zero, then g is a solution of the Thin Film equation

$$\partial_t g + \partial_x (g^n \partial_x^3 g) = 0, \quad (1.4)$$

when $n = 1$. We refer to the survey papers [1, 9] where many aspects concerning the Thin Film equation are discussed. It should be noted that similar methods to those in [7] have been used in [8] and [10] to rigorously show that, in the limit of thin fluid threads, the solutions of the moving boundary value problems for Stokes and Hele-Shaw flows converge towards the corresponding solutions (determined by the initial data) of the Thin Film equation (1.4), with $n = 3$ for Stokes and $n = 1$ for the Hele-Shaw flow. Compared with the Thin Film equation, system (1.1) is more involved because it is strongly coupled, both equations of (1.1) containing highest order derivatives of f and g , and, furthermore, there are two sources of degeneracy, because both f and g may be equal zero. Since both equations of (1.1) have fourth order, we cannot rely on maximum principles when studying problem (1.1).

Corresponding to (1.1), we define the following energy functionals

$$\mathcal{E}_1(f, g) := \frac{1}{2} \int_0^L |\partial_x f|^2 + \frac{B}{A-B} |\partial_x (f+g)|^2 dx, \quad \mathcal{E}_2(f, g) := \int_0^L \Phi(f) + B\Phi(g) dx,$$

whereby the function Φ is given by $\Phi(s) := s \ln(s) - s + 1$ for all $s \geq 0$. They will play the key role when constructing the weak solutions for the problem (1.1)-(1.3).

Using these two functionals and Galerkin approximations, we prove that the problem (1.1)-(1.3) possesses for non-negative initial data non-negative global weak solutions. To this end, we regularize first the system (1.1) and use the functional \mathcal{E}_2 to establish convergence of certain Galerkin approximations towards global weak solutions (of the regularized problem) which satisfy similar energy estimates as the classical solutions of (1.1)-(1.3). In a second step, we show that weak solutions of the regularized problem converge towards non-negative global weak solutions of the original system (1.1). The uniqueness of our weak solutions is left as an open problem (this is still an open problem also for the Thin Film equation cf. [2, 11]). We note that it has been only recently shown in [13] (see also [3, 4]), in the context of the Thin Film equation, that the non-negative weak solutions found in [2] converge exponentially fast in H^1 towards flat equilibria. In our case, this is a further open question. The second order version of (1.1), when the fluids are driven only by gravity and surface tension is neglected, has been recently investigated in [5] where existence of non-negative global weak solutions which converge exponentially fast in L_2 to flat equilibria is established (see also [7]).

In order to state our main result, we introduce now the function spaces we work with. For each $m \in \mathbb{N}$, we let $H^m := H^m((0, L))$ be the L_2 -based Sobolev space and we let H_Δ^m denote the closed subspace of H^m which has $\{\phi_k : k \in \mathbb{N}\}$ as its basis. Herein,

$$\phi_0 := \sqrt{1/L} \quad \text{and} \quad \phi_k := \sqrt{2/L} \cos(k\pi x/L), \quad k \geq 1,$$

are the normalized eigenvectors of the operator $-\partial_x^2 : H^2 \rightarrow L_2$ with zero Neumann boundary conditions. To be more precise, $f \in H_\Delta^m$ if and only if the Fourier series associated to f converges towards f in H^m . It is well-known that $H_\Delta^1 = H^1$ and, it is not difficult to see that, for $m \geq 4$, the boundary conditions (1.3) are satisfied by functions from this space.

Given $T \in (0, \infty]$, let $Q_T := (0, T) \times (0, L)$. The main result of this paper is the following theorem.

Theorem 1.1. *Let $f_0, g_0 \in H^1$ be two non-negative functions. There exist a global weak solution (f, g) of (1.1) with $(f(0), g(0)) = (f_0, g_0)$ and having the following properties:*

- (1) $f \geq 0$ and $g \geq 0$ in $(0, T) \times (0, L)$,
- (2) $f, g \in L_\infty(0, T; H^1) \cap L_2(0, T; H_\Delta^2) \cap C([0, T], C^\alpha([0, L]))$ for some arbitrary $\alpha \in (0, 1/2)$ and $\sqrt{f}\partial_x^3(Af + Bg), \sqrt{g}\partial_x^3(f + g) \in L_2(Q_T^+)$, where

$$Q_T^+ := \{(t, x) \in Q_T : (fg)(t, x) > 0\},$$

(3)

$$\begin{aligned} \int_0^L f(T)\psi \, dx - \int_0^L f_0\psi \, dx + \int_{Q_T} (A\partial_x^2 f + B\partial_x^2 g)(\partial_x f \partial_x \psi + f \partial_x^2 \psi) \, dx dt &= 0, \\ \int_0^L f(T)\psi \, dx - \int_0^L f_0\psi \, dx + \int_{Q_T} (\partial_x^2 f + \partial_x^2 g)(\partial_x g \partial_x \psi + g \partial_x^2 \psi) \, dx dt &= 0 \end{aligned}$$

for all $T > 0$ and $\psi \in H_\Delta^2$. Furthermore, the weak solutions satisfy

$$(4) \quad \|f(T)\|_{L_1} = \|f_0\|_{L_1} \quad \text{and} \quad \|g(T)\|_{L_1} = \|g_0\|_{L_1},$$

$$(5) \quad \mathcal{E}_2(f(T), g(T)) + \int_{Q_T} (A - B)|\partial_x^2 f|^2 + B|\partial_x^2(f + g)|^2 \, dx dt \leq \mathcal{E}_2(f_0, g_0)$$

for all $T \in (0, \infty)$, and

$$(6) \quad \mathcal{E}_1(f(T), g(T)) + \int_{Q_T^+} f|\partial_x^3(Af + Bg)|^2 + Bg|\partial_x^3(f + g)|^2 \, dx dt \leq \mathcal{E}_1(f_0, g_0)$$

for almost all $T \in (0, \infty)$.

We remark that since $f(t)$ and $g(t)$ belong to H_Δ^2 for almost all $t > 0$, they satisfy homogeneous Neumann boundary conditions at $x = 0$ and $x = L$ for all such t .

The outline of the paper is as follows: in Section 2 we introduce a regularized version of (1.1) and use Galerkin approximations to find, in the limit, global weak solutions of this regularized problem (see Proposition 2.1). Introducing the regularized system allows us on the one hand to use the energy functional \mathcal{E}_1 when dealing with the Galerkin approximations, and, on the other hand, to control the solutions of the regularized problem when they become negative. In Section 3 we show, by combining energy estimates for both functionals \mathcal{E}_1 and \mathcal{E}_2 , that the weak solutions of the regularized problem converge towards non-negative global weak solutions of our original problem (1.1)-(1.3).

2. THE REGULARIZED SYSTEM

In order to prove the Theorem 1.1 we shall regularize system (1.1) and use Galekin approximations to build global weak solutions for this regularized problem. These solutions are shown later on, in Section 3, to converge towards weak solutions of (1.1). To this end, given $\varepsilon \in (0, 1]$, we define the Lipschitz continuous function $a_\varepsilon : \mathbb{R} \rightarrow \mathbb{R}$ by the relation

$$a_\varepsilon(s) := \begin{cases} s + \varepsilon, & s \geq 0, \\ \varepsilon, & s < 0. \end{cases} \quad (2.1)$$

Furthermore, we define the convex function $\Phi_\varepsilon : \mathbb{R} \rightarrow \mathbb{R}$ with

$$\Phi_\varepsilon(s) := \begin{cases} (s + \varepsilon) \ln(s + \varepsilon) - (s + \varepsilon) + 1, & s \geq 0, \\ \frac{s^2}{2\varepsilon} + s \ln(\varepsilon) + \varepsilon \ln(\varepsilon) - \varepsilon + 1, & s < 0. \end{cases} \quad (2.2)$$

Since we choose $\varepsilon \leq 1$, it is easy to see that $\Phi_\varepsilon(s) \geq 0$ for all $s \in \mathbb{R}$ and that $\Phi_\varepsilon'' = 1/a_\varepsilon$. With this notation, we introduce the following regularized version of our original problem (1.1)

$$\begin{cases} \partial_t f_\varepsilon = -\partial_x [a_\varepsilon(f_\varepsilon) \partial_x^3 (A f_\varepsilon + B g_\varepsilon)], \\ \partial_t g_\varepsilon = -\partial_x [a_\varepsilon(g_\varepsilon) \partial_x^3 (f_\varepsilon + g_\varepsilon)], \end{cases} \quad (t, x) \in (0, \infty) \times \Omega. \quad (2.3)$$

Of course, this system is coupled with the initial and boundary conditions (1.2) and (1.3). Compared to (1.1), the only difference is that we replaced at one place f and g in (1.1) by $a_\varepsilon(f)$ and $a_\varepsilon(g)$, respectively, and penalize in this way the functions $f_\varepsilon, g_\varepsilon$ when they take negative values (see the definition of a_ε). Furthermore, by choosing the regularization in this way, we may still use the functional \mathcal{E}_1 to obtain useful estimates for the solutions of (2.3). For the problem consisting of (2.3) and (1.2)-(1.3) we prove the following result.

Proposition 2.1. *Let $f_0, g_0 \in H^1$ be two non-negative functions and $\varepsilon \in (0, 1]$. There exist globally defined functions f_ε and g_ε with $f_\varepsilon(0) = f_0$, $g_\varepsilon(0) = g_0$ and having the following properties:*

(i) *Given $T > 0$, the functions*

$$f_\varepsilon, g_\varepsilon \in L_\infty([0, T], H^1) \cap L_2(0, T; H_\Delta^3) \cap C([0, T], C^\alpha([0, L]))$$

for some arbitrary $\alpha \in (0, 1/2)$.

(ii) *For all $T > 0$ and $\psi \in H^1$ we have*

$$\begin{aligned} \int_0^L f_\varepsilon(T) \psi \, dx - \int_0^L f_0 \psi \, dx &= \int_{Q_T} a_\varepsilon(f_\varepsilon) \partial_x^3 (A f_\varepsilon + B g_\varepsilon) \partial_x \psi \, dx dt, \\ \int_0^L g_\varepsilon(T) \psi \, dx - \int_0^L g_0 \psi \, dx &= \int_{Q_T} a_\varepsilon(g_\varepsilon) \partial_x^3 (f_\varepsilon + g_\varepsilon) \partial_x \psi \, dx dt. \end{aligned}$$

(iii) The following energy estimates are satisfied:

$$(a) \quad \int_0^L f_\varepsilon(T) dx = \int_0^L f_0 dx \quad \text{and} \quad \int_0^L g_\varepsilon(T) dx = \int_0^L g_0 dx,$$

$$(b) \quad \int_0^L \Phi_\varepsilon(f_\varepsilon(T)) + B\Phi_\varepsilon(g_\varepsilon(T)) dx \\ + \int_{Q_T} (A - B)|\partial_x^2 f_\varepsilon|^2 + B|\partial_x^2(f_\varepsilon + g_\varepsilon)|^2 dx dt \\ \leq \int_0^L \Phi_\varepsilon(f_0) + B\Phi_\varepsilon(g_0) dx$$

for all $T \in [0, \infty)$, and

$$(c) \quad \mathcal{E}_1(f_\varepsilon(T), g_\varepsilon(T)) \\ + \frac{1}{A - B} \int_{Q_T} a_\varepsilon(f_\varepsilon)|\partial_x^3(Af_\varepsilon + Bg_\varepsilon)|^2 + Ba_\varepsilon(g_\varepsilon)|\partial_x^3(f_\varepsilon + g_\varepsilon)|^2 dx dt \\ \leq \mathcal{E}_1(f_0, g_0) \quad \text{for almost all } T \in (0, \infty).$$

We will construct the global solutions of (2.3) by using Galerkin's method. In a first step we will find, by using the Picard-Lindelöf theorem, Galerkin approximations for the solutions of (2.3) which are defined on a positive time interval. Using the energy functional \mathcal{E}_1 , we show then that in fact the approximations are defined globally. In a second step, we prove that the Galerkin approximation converge towards global solutions of the regularized system which satisfy energy inequalities for both energy functionals \mathcal{E}_1 and \mathcal{E}_2 . Though f_0 and g_0 are non-negative, it is not clear whether f_ε and g_ε preserve this property in time. However, we will show later on, in Section 3, that, for $\varepsilon \rightarrow 0$, f_ε and g_ε converge uniformly to non-negative functions.

2.1. Global existence of the Galerkin approximations.

Given f_0, g_0 in H^1 , the initial conditions of (1.1), we consider their expansions

$$f_0 = \sum_{k=0}^{\infty} f_{0k} \phi_k, \quad g_0 = \sum_{k=0}^{\infty} g_{0k} \phi_k \quad \text{in } H^1,$$

and, for each $n \in \mathbb{N}$, the partial sums

$$f_0^n := \sum_{k=0}^n f_{0k} \phi_k, \quad g_0^n := \sum_{k=0}^n g_{0k} \phi_k.$$

We first seek continuously differentiable functions

$$f_\varepsilon^n := \sum_{k=0}^n F_\varepsilon^k(t) \phi_k, \quad g_\varepsilon^n := \sum_{k=0}^n G_\varepsilon^k(t) \phi_k$$

which solve (2.3) when testing with functions from the vector space $\langle \phi_0, \dots, \phi_n \rangle$, and additionally

$$f_\varepsilon^n(0) = f_0^n, \quad g_\varepsilon^n(0) = g_0^n.$$

By construction, the functions $(f_\varepsilon^n, g_\varepsilon^n)$ satisfy the boundary conditions (1.3) and, if we test (2.3) with constant functions, it follows at once that necessarily F_ε^0 and G_ε^0 are constant functions

$$F_\varepsilon^0(t) = f_{00}, \quad G_\varepsilon^0(t) = g_{00}, \quad t \geq 0. \quad (2.4)$$

Moreover, the tuple $(\vec{F}_\varepsilon^n, \vec{G}_\varepsilon^n) := (F_\varepsilon^1, \dots, F_\varepsilon^n, G_\varepsilon^1, \dots, G_\varepsilon^n)$ is the solution of the initial value problem

$$(\vec{F}_\varepsilon^n, \vec{G}_\varepsilon^n)' = \Psi(\vec{F}_\varepsilon^n, \vec{G}_\varepsilon^n), \quad (\vec{F}_\varepsilon^n, \vec{G}_\varepsilon^n)(0) = (f_{01}, \dots, f_{0n}, g_{01}, \dots, g_{0n}), \quad (2.5)$$

where $\Psi := (\Psi_1, \Psi_2) : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ is given by

$$\begin{aligned} \Psi_{1,j}(x, y) &= \sum_{k=1}^n (Ax_k + By_k) \int_0^L a_\varepsilon \left(f_{00}\phi_0 + \sum_{l=1}^n x_l\phi_l \right) \partial_x^3 \phi_k \partial_x \phi_j \, dx \\ \Psi_{2,j}(x, y) &= \sum_{k=1}^n (x_k + y_k) \int_0^L a_\varepsilon \left(g_{00}\phi_0 + \sum_{l=1}^n y_l\phi_l \right) \partial_x^3 \phi_k \partial_x \phi_j \, dx, \end{aligned}$$

for all $x, y \in \mathbb{R}^n$. Since a_ε is Lipschitz continuous, we deduce that Ψ is locally Lipschitz continuous on \mathbb{R}^{2n} , and therefore problem (2.5) possesses a unique solution $(\vec{F}_\varepsilon^n, \vec{G}_\varepsilon^n)$ defined on a maximal interval $[0, T_\varepsilon^n)$. In order to prove that the solution is global, that is $T_\varepsilon^n = \infty$ for all $\varepsilon \in (0, 1]$ and $n \in \mathbb{N}$, we make use of the energy functional \mathcal{E}_1 . Indeed, since $\partial_x^2 f_\varepsilon^n, \partial_x^2 g_\varepsilon^n \in \langle \phi_0, \dots, \phi_n \rangle$, we may use them as test functions for (2.3). Integrating by parts, we then get the following relation

$$\begin{aligned} & \frac{d}{dt} \mathcal{E}_1(f_\varepsilon^n, g_\varepsilon^n) \\ &= \frac{1}{A-B} \int_0^L A \partial_x f_\varepsilon^n \partial_t (\partial_x f_\varepsilon^n) + B \partial_x f_\varepsilon^n \partial_t (\partial_x g_\varepsilon^n) \\ & \quad + B \partial_x g_\varepsilon^n \partial_t (\partial_x f_\varepsilon^n) + B \partial_x g_\varepsilon^n \partial_t (\partial_x g_\varepsilon^n) \, dx \\ &= -\frac{1}{A-B} \int_0^L A \partial_x^2 f_\varepsilon^n \partial_t f_\varepsilon^n + B [\partial_x^2 f_\varepsilon^n \partial_t g_\varepsilon^n \, dx + \partial_x^2 g_\varepsilon^n \partial_t g_\varepsilon^n + \partial_x^2 g_\varepsilon^n \partial_t f_\varepsilon^n] \, dx \\ &= -\frac{1}{A-B} \int_0^L [A a_\varepsilon(f_\varepsilon^n) \partial_x^3 f_\varepsilon^n \partial_x^3 (A f_\varepsilon^n + B g_\varepsilon^n) + B a_\varepsilon(g_\varepsilon^n) \partial_x^3 f_\varepsilon^n \partial_x^3 (f_\varepsilon^n + g_\varepsilon^n) \\ & \quad + B a_\varepsilon(f_\varepsilon^n) \partial_x^3 g_\varepsilon^n \partial_x^3 (A f_\varepsilon^n + B g_\varepsilon^n) + B a_\varepsilon(g_\varepsilon^n) \partial_x^3 g_\varepsilon^n \partial_x^3 (f_\varepsilon^n + g_\varepsilon^n)] \, dx, \end{aligned} \quad (2.6)$$

and taking into account that $\mathcal{E}_1(f_0^n, g_0^n) \leq \mathcal{E}_1(f_0, g_0)$ for all $n \in \mathbb{N}$, we find after integrating with respect to time that

$$\begin{aligned} \mathcal{E}_1(f_\varepsilon^n(T), g_\varepsilon^n(T)) &+ \frac{1}{A-B} \int_{Q_T} a_\varepsilon(f_\varepsilon^n) |\partial_x^3 (A f_\varepsilon^n + B g_\varepsilon^n)|^2 \\ &+ B a_\varepsilon(g_\varepsilon^n) |\partial_x^3 (f_\varepsilon^n + g_\varepsilon^n)|^2 \, dx \, dt \leq \mathcal{E}_1(f_0, g_0) \end{aligned} \quad (2.7)$$

for all $T > 0$. Whence, there exists a positive constant C , which is independent of time, such that $|(\vec{F}_\varepsilon^n(T), \vec{G}_\varepsilon^n(T))| < C$ for all $T < T_\varepsilon^n$. Together with (2.4), we conclude that for each $n \in \mathbb{N}$ and $\varepsilon \in (0, 1]$, the Galerkin approximations $(f_\varepsilon^n, g_\varepsilon^n)$ are defined globally.

2.2. Convergence of the Galerkin approximations.

Let $T > 0$ and $\varepsilon \in (0, 1]$ be fixed. From the energy estimate (2.7) we deduce that

$$\partial_x f_\varepsilon^n, \partial_x g_\varepsilon^n \text{ are bounded in } L_\infty(0, T; L_2), \quad (2.8)$$

$$\sqrt{a_\varepsilon(f_\varepsilon^n)} \partial_x^3 (A f_\varepsilon^n + B g_\varepsilon^n), \sqrt{a_\varepsilon(g_\varepsilon^n)} \partial_x^3 (f_\varepsilon^n + g_\varepsilon^n) \text{ are bounded in } L_2(Q_T), \quad (2.9)$$

uniformly in $n \in \mathbb{N}$ and $\varepsilon \in (0, 1]$. In view of $a_\varepsilon \geq \varepsilon$ and $A > B$, we obtain from (2.9) that

$$\partial_x^3 f_\varepsilon^n, \partial_x^3 g_\varepsilon^n \text{ are bounded in } L_2(Q_T), \quad (2.10)$$

uniformly in $n \in \mathbb{N}$. Furthermore, by virtue of (2.4), we see that the mass of both fluids is preserved by the Galerkin approximations

$$\int_0^L f_\varepsilon^n(t) dx = \int_0^L f_0 dx \quad \text{and} \quad \int_0^L g_\varepsilon^n(t) dx = \int_0^L g_0 dx \quad \text{for all } t \in [0, T]. \quad (2.11)$$

Invoking now (2.8), (2.11), and the Poincaré-Wirtinger inequality we conclude that in fact

$$f_\varepsilon^n, g_\varepsilon^n \text{ are bounded in } L_\infty(0, T; H^1) \text{ uniformly in } \varepsilon \in (0, 1] \text{ and } n \in \mathbb{N}, \quad (2.12)$$

while, owing to (2.10) and (2.11), the same inequality implies

$$f_\varepsilon^n, g_\varepsilon^n \text{ are bounded in } L_2(0, T; H^3) \text{ uniformly in } n. \quad (2.13)$$

We consider now the partial derivatives with respect to time, and observe that the first equation of (2.3) can be written in the more compact form $\partial_t f_\varepsilon^n = -\partial_x H_\varepsilon^n$ where, by (2.9), (2.12), and using the embedding $H^1 \hookrightarrow L_\infty$, the right-hand side $H_\varepsilon^n := a(f_\varepsilon^n)(A \partial_x^3 f_\varepsilon^n + B \partial_x^3 g_\varepsilon^n)$ is bounded in $L_2(Q_T)$ uniformly in ε and n . Therefore, given $\zeta \in H^1$, we set

$$\zeta^n := \sum_{k=0}^n (\zeta | \phi_k) \phi_k$$

and, using integration by parts, obtain

$$\begin{aligned} |(\partial_t f_\varepsilon^n(t) | \zeta)| &= |(\partial_t f_\varepsilon^n(t) | \zeta^n)| \\ &= |(H_\varepsilon^n | \partial_x \zeta^n)| \\ &\leq \|H_\varepsilon^n\|_{L_2(Q_T)} \|\zeta^n\|_{H^1} \\ &\leq \|H_\varepsilon^n\|_{L_2(Q_T)} \|\zeta\|_{H^1}. \end{aligned}$$

This means that

$$\partial_t f_\varepsilon^n, \partial_t g_\varepsilon^n \text{ are bounded in } L_2(0, T; (H^1)') \text{ uniformly in } \varepsilon \text{ and } n. \quad (2.14)$$

Gathering (2.12)-(2.14), we obtain from Corollary 4 in [12], by making also use of the embeddings

$$H^1 \xrightarrow{comp.} C^\alpha([0, L]) \hookrightarrow (H^1)' \quad \text{and} \quad H^3 \xrightarrow{comp.} C^{2+\alpha}([0, L]) \hookrightarrow (H^1)'$$

for $\alpha \in [0, 1/2)$, that

$$f_\varepsilon^n, g_\varepsilon^n \text{ are relatively compact in } C([0, T], C^\alpha([0, L])) \cap L_2(0, T; C^{2+\alpha}([0, L])).$$

Whence, for each $\varepsilon \in (0, 1]$, there exist functions

$$f_\varepsilon, g_\varepsilon \in C([0, T], C^\alpha([0, L])) \cap L_2(0, T; C^{2+\alpha}([0, L]))$$

and subsequences of (f_ε^n) and (g_ε^n) (which we denote again by (f_ε^n) and (g_ε^n)) such that

$$f_\varepsilon^n \rightarrow f_\varepsilon \quad \text{and} \quad g_\varepsilon^n \rightarrow g_\varepsilon \quad \text{in } C([0, T], C^\alpha([0, L])) \cap L_2(0, T; C^{2+\alpha}([0, L])). \quad (2.15)$$

Moreover, we deduce from (2.13) that

$$\partial_x^p f_\varepsilon^n \rightharpoonup \partial_x^p f_\varepsilon \quad \text{and} \quad \partial_x^p g_\varepsilon^n \rightharpoonup \partial_x^p g_\varepsilon \quad \text{in } L_2(Q_T) \text{ for } p = 1, 2, 3, \quad (2.16)$$

and therefore $f_\varepsilon, g_\varepsilon \in L_2([0, T], H^3)$. Additionally, since $f_\varepsilon^n(t), g_\varepsilon^n(t) \in H_\Delta^3$, we get, by virtue of (2.15), that $\partial_x f_\varepsilon = \partial_x g_\varepsilon = 0$ at $x = 0, L$ for almost all $t \in [0, T]$, which yields $f_\varepsilon, g_\varepsilon \in L_2([0, T], H_\Delta^3)$.

2.3. Proof of Proposition 2.1.

First of all, $f_\varepsilon^n(0) = f_0^n$ for all $n \in \mathbb{N}$ and since $f_0 \in H^1$ we conclude that $f_\varepsilon(0) = f_0$ for all $\varepsilon \in (0, 1]$. Similarly, we have $g_\varepsilon(0) = g_0$ for all $\varepsilon \in (0, 1]$. Furthermore, it is clear from (2.11) and (2.15) that the weak solutions $(f_\varepsilon, g_\varepsilon)$ satisfy the relation (iii)(a) of Proposition 2.1.

We pass now to the limit in the energy estimate (2.7). By virtue of (2.9), (2.15), and (2.16) we have

$$\begin{aligned} \sqrt{a_\varepsilon(f_\varepsilon^n)} \partial_x^3 (A f_\varepsilon^n + B g_\varepsilon^n) &\rightharpoonup \sqrt{a_\varepsilon(f_\varepsilon)} \partial_x^3 (A f_\varepsilon + B g_\varepsilon), \\ \sqrt{a_\varepsilon(g_\varepsilon^n)} \partial_x^3 (f_\varepsilon^n + g_\varepsilon^n) &\rightharpoonup \sqrt{a_\varepsilon(g_\varepsilon)} \partial_x^3 (f_\varepsilon + g_\varepsilon) \end{aligned} \quad \text{in } L_2(Q_T).$$

Furthermore, by (2.15) we know that $f_\varepsilon^n(t) \rightarrow f_\varepsilon(t)$ in H^1 for almost all $t \in [0, T]$, so that, by passing to the limit $n \rightarrow \infty$ in (2.7), we obtain the estimate (iii)(c) of Proposition 2.1.

Claim (i) of Proposition 2.1 is now a simple consequence of the assertions (iii)(a) and (iii)(c) of the same proposition.

We now prove the assertion (ii) of Proposition 2.1. To this end, we pick an arbitrary function $\psi \in H^1$ and, testing (2.3) with $\psi^n := \sum_{k=0}^n (\psi | \phi_k) \phi_k$, we obtain the following relations

$$\begin{aligned} \int_0^L f_\varepsilon^n(T) \psi^n dx - \int_0^L f_{0n} \psi^n dx &= \int_{Q_T} a_\varepsilon(f_\varepsilon^n) \partial_x^3 (A f_\varepsilon^n + B g_\varepsilon^n) \partial_x \psi^n dx dt, \\ \int_0^L g_\varepsilon^n(T) \psi^n dx - \int_0^L g_{0n} \psi^n dx &= \int_{Q_T} a_\varepsilon(g_\varepsilon^n) \partial_x^3 (f_\varepsilon^n + g_\varepsilon^n) \partial_x \psi^n dx dt. \end{aligned}$$

Invoking (2.9) and (2.12), we see that $a_\varepsilon(f_\varepsilon^n) \partial_x^3 (A f_\varepsilon^n + B g_\varepsilon^n)$ and $a_\varepsilon(g_\varepsilon^n) \partial_x^3 (f_\varepsilon^n + g_\varepsilon^n)$ are bounded in $L_2(Q_T)$ uniformly in ε and n . Using (2.15) and (2.16), we may even identify their weak limit

$$\begin{aligned} a_\varepsilon(f_\varepsilon^n) \partial_x^3 (A f_\varepsilon^n + B g_\varepsilon^n) &\rightharpoonup a_\varepsilon(f_\varepsilon) \partial_x^3 (A f_\varepsilon + B g_\varepsilon), \\ a_\varepsilon(g_\varepsilon^n) \partial_x^3 (f_\varepsilon^n + g_\varepsilon^n) &\rightharpoonup a_\varepsilon(g_\varepsilon) \partial_x^3 (f_\varepsilon + g_\varepsilon) \end{aligned} \quad \text{in } L_2(Q_T), \quad (2.17)$$

and the assertion (ii) of Proposition 2.1 follows from the previous identities when letting $n \rightarrow \infty$.

We end this paragraph with the proof of the estimate (iii)(b) of Proposition 2.1. Let us observe that $\Phi'_\varepsilon(f_\varepsilon^n(t))$ and $\Phi'_\varepsilon(f_\varepsilon(t))$ belong to H^1 for almost all $t \in [0, T]$, meaning that

$$\Phi'_\varepsilon(f_\varepsilon^n(t)) = \sum_{k=0}^{\infty} (\Phi'_\varepsilon(f_\varepsilon^n(t)) | \phi_k) \phi_k, \quad \Phi'_\varepsilon(f_\varepsilon(t)) = \sum_{k=0}^{\infty} (\Phi'_\varepsilon(f_\varepsilon(t)) | \phi_k) \phi_k \quad \text{in } H^1 \quad (2.18)$$

for almost all $t \in [0, T]$. Of course, (2.18) is also valid when replacing f by g . In view of (2.18), we obtain the following relations

$$\begin{aligned} \frac{d}{dt} \int_0^L \Phi_\varepsilon(f_\varepsilon^n) + B\Phi_\varepsilon(g_\varepsilon^n) dx &= \int_0^L \Phi'_\varepsilon(f_\varepsilon^n) \partial_t f_\varepsilon^n + B\Phi'_\varepsilon(g_\varepsilon^n) \partial_t g_\varepsilon^n dx \\ &= \int_0^L a_\varepsilon(f_\varepsilon^n) \partial_x^3 (A f_\varepsilon^n + B g_\varepsilon^n) \sum_{k=0}^n (\Phi'_\varepsilon(f_\varepsilon^n) | \phi_k) \partial_x \phi_k \\ &\quad + B a_\varepsilon(g_\varepsilon^n) \partial_x^3 (f_\varepsilon^n + g_\varepsilon^n) \sum_{k=0}^n (\Phi'_\varepsilon(g_\varepsilon^n) | \phi_k) \partial_x \phi_k dx, \end{aligned} \quad (2.19)$$

and, integrating with respect to time, we arrive at

$$\begin{aligned} \int_0^L \Phi_\varepsilon(f_\varepsilon^n(T)) + B\Phi_\varepsilon(g_\varepsilon^n(T)) dx &= \int_{Q_T} \left[a_\varepsilon(f_\varepsilon^n) (A \partial_x^3 f_\varepsilon^n + B \partial_x^3 g_\varepsilon^n) \sum_{k=0}^n (\Phi'_\varepsilon(f_\varepsilon^n) | \phi_k) \partial_x \phi_k \right. \\ &\quad \left. + B a_\varepsilon(g_\varepsilon^n) (\partial_x^3 f_\varepsilon^n + \partial_x^3 g_\varepsilon^n) \sum_{k=0}^n (\Phi'_\varepsilon(g_\varepsilon^n) | \phi_k) \partial_x \phi_k \right] dx dt \\ &\quad + \int_0^L \Phi_\varepsilon(f_0^n) + B\Phi_\varepsilon(g_0^n) dx. \end{aligned} \quad (2.20)$$

In order to pass to the limit $n \rightarrow \infty$ in relation (2.20) we have to determine what happens with the two integrals on the right-hand side of (2.20). Using (2.18), we have

$$\begin{aligned} &\left\| \sum_{k=0}^n (\Phi'_\varepsilon(f_\varepsilon^n) | \phi_k) \partial_x \phi_k - \Phi''_\varepsilon(f_\varepsilon) \partial_x f_\varepsilon \right\|_{L_2(Q_T)}^2 \\ &\leq 2 \left\| \sum_{k=0}^n (\Phi'_\varepsilon(f_\varepsilon^n) - \Phi'_\varepsilon(f_\varepsilon) | \phi_k) \partial_x \phi_k \right\|_{L_2(Q_T)}^2 \\ &\quad + 2 \left\| \sum_{k=0}^n (\Phi'_\varepsilon(f_\varepsilon) | \phi_k) \partial_x \phi_k - \Phi''_\varepsilon(f_\varepsilon) \partial_x f_\varepsilon \right\|_{L_2(Q_T)}^2. \end{aligned}$$

Taking into account that the first sum on the right-hand side of the latter inequality is the truncation of the Fourier series of $\Phi''_\varepsilon(f_\varepsilon^n)\partial_x f_\varepsilon^n - \Phi''_\varepsilon(f_\varepsilon)\partial_x f_\varepsilon$, cf. (2.18), its norm may be estimated as follows

$$\begin{aligned}
\left\| \sum_{k=0}^n (\Phi'_\varepsilon(f_\varepsilon^n) - \Phi'_\varepsilon(f_\varepsilon))\phi_k \partial_x \phi_k \right\|_{L_2(Q_T)}^2 &\leq \|\Phi''_\varepsilon(f_\varepsilon^n)\partial_x f_\varepsilon^n - \Phi''_\varepsilon(f_\varepsilon)\partial_x f_\varepsilon\|_{L_2(Q_T)}^2 \\
&\leq 2\|\Phi''_\varepsilon(f_\varepsilon^n) - \Phi''_\varepsilon(f_\varepsilon)\|_{L_\infty(Q_T)}^2 \|\partial_x f_\varepsilon\|_{L_2(Q_T)}^2 \\
&\quad + 2\|\Phi''_\varepsilon(f_\varepsilon^n)\|_{L_\infty(Q_T)}^2 \|\partial_x f_\varepsilon^n - \partial_x f_\varepsilon\|_{L_2(Q_T)}^2 \\
&\leq 2\varepsilon^{-4} \|f_\varepsilon^n - f_\varepsilon\|_{L_\infty(Q_T)}^2 \|\partial_x f_\varepsilon\|_{L_2(Q_T)}^2 \\
&\quad + 2\varepsilon^{-2} \|\partial_x f_\varepsilon^n - \partial_x f_\varepsilon\|_{L_2(Q_T)}^2.
\end{aligned}$$

We note that the last inequality has been obtained by using the fact that Φ''_ε is Lipschitz continuous with Lipschitz constant ε^{-2} and $0 \leq \Phi''_\varepsilon \leq \varepsilon^{-1}$, properties which readily follow from (2.1), (2.2), and the relation $\Phi''_\varepsilon = 1/a_\varepsilon$. Invoking (2.15), we resume our calculation with

$$\left\| \sum_{k=0}^n (\Phi'_\varepsilon(f_\varepsilon^n) - \Phi'_\varepsilon(f_\varepsilon))\phi_k \partial_x \phi_k \right\|_{L_2(Q_T)}^2 \rightarrow_{n \rightarrow \infty} 0. \quad (2.21)$$

Concerning the second term, we obtain from (2.18) that

$$\left\| \sum_{k=0}^n (\Phi'_\varepsilon(f_\varepsilon))\phi_k \partial_x \phi_k - \Phi''_\varepsilon(f_\varepsilon)\partial_x f_\varepsilon \right\|_{L_2} = \left\| \sum_{k=n+1}^\infty (\Phi'_\varepsilon(f_\varepsilon))\phi_k \partial_x \phi_k \right\|_{L_2} \searrow_{n \rightarrow \infty} 0$$

for almost all $t \in [0, T]$, and Lebesgue's dominated convergence theorem yields

$$\left\| \sum_{k=0}^n (\Phi'_\varepsilon(f_\varepsilon^n))\phi_k \partial_x \phi_k - \Phi''_\varepsilon(f_\varepsilon)\partial_x f_\varepsilon \right\|_{L_2(Q_T)} \rightarrow_{n \rightarrow \infty} 0. \quad (2.22)$$

Gathering (2.21) and (2.22), we conclude that

$$\sum_{k=0}^n (\Phi'_\varepsilon(f_\varepsilon^n))\phi_k \partial_x \phi_k \rightarrow_{n \rightarrow \infty} \Phi''_\varepsilon(f_\varepsilon)\partial_x f_\varepsilon \quad \text{in } L_2(Q_T). \quad (2.23)$$

Clearly, (2.23) remains true if we replace f by g . We sum (2.17), (2.23), use (2.15) and the fact that both f_0 and g_0 are non-negative to obtain from (2.20), when letting $n \rightarrow \infty$, the desired assertion (iii)(b) of Proposition 2.1.

3. THE PROOF OF THEOREM 1.1

We shall use the global weak solutions $(f_\varepsilon, g_\varepsilon)$ of the regularized problem (2.3) to find, in the limit $\varepsilon \rightarrow 0$, global weak solutions of our original system (1.1). The key role is now played by the second energy functional \mathcal{E}_2 , which will be used to prove that the weak solutions we obtain are non-negative and to identify in $L_2(0, T; H^2)$ a weak limit of the global solutions of (2.3). Using integration by parts, we may eliminate then from the right-hand side of (ii) Proposition 2.1 the third order derivatives of f_ε

and g_ε , for which we don't have any kind of uniform bounds, and obtain in the limit $\varepsilon \rightarrow 0$ the assertion (3) of Theorem 1.1.

To do so, we collect first some estimates for the family $(f_\varepsilon, g_\varepsilon)$ which have been already established in Section 2. We have to pay attention because some of the estimates proven before are uniform only with respect to n , and of no use in this final part. Invoking (2.9) and (2.12), we deduce the uniform boundedness of

$$f_\varepsilon, g_\varepsilon \quad \text{in } L_\infty(0, T; H^1), \quad (3.1)$$

$$\sqrt{a_\varepsilon(f_\varepsilon)} \partial_x^3 (A f_\varepsilon + B g_\varepsilon), \sqrt{a_\varepsilon(g_\varepsilon)} \partial_x^3 (f_\varepsilon + g_\varepsilon) \quad \text{in } L_2(Q_T), \quad (3.2)$$

while, by virtue of Proposition (iii)(a) and (b),

$$\partial_x^2 f_\varepsilon, \partial_x^2 g_\varepsilon \quad \text{are uniformly bounded in } L_2(Q_T), \quad (3.3)$$

$$\int_0^L f_\varepsilon(T) dx = \int_0^L f_0 dx, \quad \int_0^L g_\varepsilon(T) dx = \int_0^L g_0 dx, \quad (3.4)$$

for all $T > 0$. Lastly, we observe that the estimates (2.12) and (2.14) are both uniform with respect to $\varepsilon \in (0, 1]$ and $n \in \mathbb{N}$. This implies that the families $\{f_\varepsilon^n : \varepsilon \in (0, 1], n \in \mathbb{N}\}$ and $\{g_\varepsilon^n : \varepsilon \in (0, 1], n \in \mathbb{N}\}$ are both relatively compact in $C([0, T], C^\alpha([0, L]))$, if $\alpha \in [0, 1/2)$, and therefore

$$(f_\varepsilon), (g_\varepsilon) \quad \text{are relatively compact in } C([0, T], C^\alpha([0, L])). \quad (3.5)$$

Consequently, there exist subsequences (f_{ε_k}) and (g_{ε_k}) and functions f, g such that

$$f_{\varepsilon_k} \rightarrow f \quad \text{and} \quad g_{\varepsilon_k} \rightarrow g \quad \text{in } C([0, T], C^\alpha([0, L])), \quad (3.6)$$

while, owing to (3.1), (3.3), we conclude that $f_\varepsilon, g_\varepsilon$ are bounded in $L_2(0, T; H^2)$, which ensures, after possibly extracting further subsequences, weak convergence in $L_2(Q_T)$ of the spatial derivatives up to order 2

$$\partial_x^p f_{\varepsilon_k} \rightharpoonup \partial_x^p f \quad \text{and} \quad \partial_x^p g_{\varepsilon_k} \rightharpoonup \partial_x^p g \quad \text{in } L_2(Q_T) \text{ for } p = 1, 2. \quad (3.7)$$

Recalling Proposition 2.1 (i) and (3.6), we deduce that $f, g \in L_2(0, T; H_\Delta^2)$ for all $T > 0$. Moreover, the sequences (f_{ε_k}) and (g_{ε_k}) converge strongly towards f and g , respectively, in a different norm than in (3.6).

Lemma 3.1. *Given $T > 0$, we have:*

$$f_{\varepsilon_k} \rightarrow f \quad \text{and} \quad g_{\varepsilon_k} \rightarrow g \quad \text{in } L_4(0, T; H^1). \quad (3.8)$$

Proof. We prove only the assertion for f . Since $f(t)$ and $f_{\varepsilon_k}(t)$ belong to H_Δ^2 for almost all $t \in [0, T]$, we conclude that their first order derivatives at 0 and L must vanish. Whence, using integration by parts, we get

$$\begin{aligned} \int_0^T \left(\int_0^L |\partial_x(f_{\varepsilon_k} - f)|^2 dx \right)^2 dt &= \int_0^T \left(\int_0^L \partial_x^2(f_{\varepsilon_k} - f)(f_{\varepsilon_k} - f) dx \right)^2 dt \\ &\leq \int_0^T \|\partial_x^2(f_{\varepsilon_k} - f)\|_{L_2}^2 \|f_{\varepsilon_k} - f\|_{L_2}^2 dt \leq L \|f_{\varepsilon_k} - f\|_{L_\infty(Q_T)}^2 \|\partial_x^2(f_{\varepsilon_k} - f)\|_{L_2(Q_T)}^2, \end{aligned}$$

and, together with (3.3) and (3.6), we get the desired conclusion. \square

Particularly, we obtain from (3.8), that $f_{\varepsilon_k}(t) \rightarrow f(t)$ and $g_{\varepsilon_k}(t) \rightarrow g(t)$ in H^1 for almost all $t \in [0, T]$, and together with the estimate (3.1) we conclude that $f, g \in L_\infty(0, T; H^1)$. Furthermore, $f_\varepsilon(0) = f_0$ and $g_\varepsilon(0) = g_0$ for all $\varepsilon \in (0, 1]$, so that (3.6) yields $f(0) = f_0$ and $g(0) = g_0$. The estimate (4) of Theorem 1.1 follows by combining (3.6), the assertion (iii)(a) of Proposition 2.1, and Lemma 3.2 below.

We use now the energy estimate (iii)(b) of Proposition 2.1, to establish the assertion (1) of our main result Theorem 1.1.

Lemma 3.2. *The functions f and g found above are non-negative.*

Proof. Assume that there exists $(T, x_0) \in Q_\infty$ such that $f(T, x_0) < 0$. Since by (3.6) $f_{\varepsilon_k} \rightarrow f$ in $C(\overline{Q}_T)$, we conclude that there exists a constant $\delta > 0$ and $k_0 \in \mathbb{N}$ with the property that $f_{\varepsilon_k}(T, x) < -\delta$ for all $x \in [0, L]$ with $|x - x_0| < \delta$ and all $k \geq k_0$. We then infer from (2.2) that

$$\Phi_{\varepsilon_k}(f_{\varepsilon_k}(T, x)) = \frac{f_{\varepsilon_k}^2(T, x)}{2\varepsilon_k} + f_{\varepsilon_k}(T, x) \ln(\varepsilon_k) + \varepsilon_k \ln(\varepsilon_k) - \varepsilon_k + 1 \geq \frac{\delta^2}{2\varepsilon_k}$$

for all x and k as above. This contradicts the assertion (iii)(b) of Proposition 2.1. Clearly, the argument is true when replacing f by g , and this proves the claim. \square

In order to deduce the energy estimate Theorem 1.1 (5), we recall (2.2) and notice that, for all $k \in \mathbb{N}$, we have $\Phi_{\varepsilon_k}(f_{\varepsilon_k}) \geq \tilde{\Phi}_{\varepsilon_k}(f_{\varepsilon_k})$, where

$$\tilde{\Phi}_{\varepsilon_k}(s) := \begin{cases} (s + \varepsilon_k) \ln(s + \varepsilon_k) - (s + \varepsilon_k) + 1, & s \geq 0, \\ \varepsilon_k \ln(\varepsilon_k) - \varepsilon_k + 1, & s < 0. \end{cases}$$

Given $t \in [0, T]$, the sequence $(\tilde{\Phi}_{\varepsilon_k}(f_{\varepsilon_k}(t)))$ is bounded in $C([0, L])$ and $\tilde{\Phi}_{\varepsilon_k}(f_{\varepsilon_k}(t)) \rightarrow \Phi(f(t))$ pointwise on $[0, L]$. Lebesgue's dominated convergence implies then

$$\liminf_{k \rightarrow \infty} \int_0^L \Phi_{\varepsilon_k}(f_{\varepsilon_k}(T)) dx \geq \liminf_{k \rightarrow \infty} \int_0^L \tilde{\Phi}_{\varepsilon_k}(f_{\varepsilon_k}(T)) dx = \int_0^L \Phi(f(T)) dx. \quad (3.9)$$

Of course, the relation still remains true when replacing f by g . By virtue of (3.7), we may pass to $\liminf_{k \rightarrow \infty}$ in relation (iii)(b) of Proposition 2.1, and obtain in this way the desired energy estimate (5) of Theorem 1.1.

To deal with the energy estimate (6) of Theorem 1.1, we observe first that for all $k \in \mathbb{N}$

$$|a_{\varepsilon_k}(f_{\varepsilon_k}) - f| \leq \varepsilon_k + |f_{\varepsilon_k} - f|,$$

meaning, by (3.6), that

$$a_{\varepsilon_k}(f_{\varepsilon_k}) \rightarrow f \quad \text{and} \quad a_{\varepsilon_k}(g_{\varepsilon_k}) \rightarrow g \quad \text{in } C(\overline{Q}_T). \quad (3.10)$$

For every positive integer m , we introduce now the set

$$Q_T^m := \{(t, x) \in Q_T : f(t, x) > m^{-1} \text{ and } g(t, x) > m^{-1}\},$$

where we may control, by virtue of the estimate (iii)(c) of Proposition 2.1 and (3.10), the third order derivatives of both f_{ε_k} and g_{ε_k} :

$$(\partial_x^3 f_{\varepsilon_k}), (\partial_x^3 g_{\varepsilon_k}) \quad \text{are uniformly bounded in } L_2(Q_T^m). \quad (3.11)$$

Taking into account that $Q_T^+ = \cup_m Q_T^m$, we may assume, after possibly extracting a further subsequence, that

$$\partial_x^3 f_{\varepsilon_k} \rightharpoonup \partial_x^3 f, \quad \partial_x^3 g_{\varepsilon_k} \rightharpoonup \partial_x^3 g \quad \text{in } L_2(Q_T^m)$$

for all $m \in \mathbb{N}$, which, together with (3.10), implies

$$\begin{aligned} \sqrt{a_{\varepsilon_k}(f_{\varepsilon_k})} \partial_x^3 (A f_{\varepsilon_k} + B g_{\varepsilon_k}) &\rightharpoonup \sqrt{f} \partial_x^3 (A f + B g), \\ \sqrt{a_{\varepsilon_k}(g_{\varepsilon_k})} \partial_x^3 (f_{\varepsilon_k} + g_{\varepsilon_k}) &\rightharpoonup \sqrt{g} \partial_x^3 (f + g) \end{aligned} \quad \text{in } L_1(Q_T^m). \quad (3.12)$$

In fact, by virtue of (3.2), the weak convergence in (3.12) takes place in $L_2(Q_T^m)$. Recalling Proposition 2.1 (iii)(c) and Lemma 3.1, for $k \rightarrow \infty$, we obtain the desired estimates (2) and (6) of Theorem 1.1.

In order to complete the proof of Theorem 1.1, we are left to prove the relations (3). To this end, we pick $\psi \in H_{\Delta}^2$. Since a_{ε_k} is Lipschitz continuous, we obtain from Proposition 2.1 (i) that $a_{\varepsilon_k}(f_{\varepsilon_k}(t)) \in H^1$ for almost all $t \in (0, T)$ and

$$\partial_x(a_{\varepsilon_k}(f_{\varepsilon_k}))(t, x) = \chi_{(0, \infty)}(f_{\varepsilon_k}) \partial_x f_{\varepsilon_k}, \quad \text{a.e. in } Q_T,$$

whereby $\chi_{(0, \infty)}$ denotes the characteristic function of the interval $(0, \infty)$. Integrating by parts in the first relation of Proposition 2.1 (ii), we arrive at

$$\int_0^L f_{\varepsilon_k}(T) \psi \, dx - \int_0^L f_0 \psi \, dx = \int_0^T \int_0^L a_{\varepsilon_k}(f_{\varepsilon_k}) \partial_x^3 (A f_{\varepsilon_k} + B g_{\varepsilon_k}) \partial_x \psi \, dx dt = I_{1,k} + I_{2,k} \quad (3.13)$$

where

$$\begin{aligned} I_{1,k} &:= - \int_{Q_T} \partial_x(a_{\varepsilon_k}(f_{\varepsilon_k})) \partial_x^2 (A f_{\varepsilon_k} + B g_{\varepsilon_k}) \partial_x \psi \, dx dt, \\ I_{2,k} &:= - \int_{Q_T} a_{\varepsilon_k}(f_{\varepsilon_k}) \partial_x^2 (A f_{\varepsilon_k} + B g_{\varepsilon_k}) \partial_x^2 \psi \, dx dt. \end{aligned}$$

We note the use of $\psi \in H_{\Delta}^2$ to eliminate the boundary terms in (3.13) due to $\partial_x \psi(0) = \partial_x \psi(L) = 0$.

Combining (3.7) and (3.10), we obtain for $k \rightarrow \infty$ that

$$I_{2,k} \rightarrow - \int_{Q_T} f (A \partial_x^2 f + B \partial_x^2 g) \partial_x^2 \psi \, dx dt. \quad (3.14)$$

We consider now the integral $I_{1,k}$, and notice that in order to show the relation

$$I_{1,k} \rightarrow - \int_{Q_T} \partial_x f (A \partial_x^2 f + B \partial_x^2 g) \partial_x \psi \, dx dt \quad (3.15)$$

it suffices to prove that

$$\partial_x(a_{\varepsilon_k}(f_{\varepsilon_k})) \rightarrow \partial_x f \quad \text{in } L_2(Q_T). \quad (3.16)$$

To this end, we write $\partial_x(a_{\varepsilon_k}(f_{\varepsilon_k})) - \partial_x f = (\partial_x(a_{\varepsilon_k}(f_{\varepsilon_k})) - \partial_x f_{\varepsilon_k}) + (\partial_x f_{\varepsilon_k} - \partial_x f)$, and conclude from Lemma 3.1 that $(\partial_x f_{\varepsilon_k} - \partial_x f) \rightarrow 0$ in $L_2(Q_T)$. Furthermore, the first term may be written as $(\partial_x(a_{\varepsilon_k}(f_{\varepsilon_k})) - \partial_x f_{\varepsilon_k}) = (\chi_{(0, \infty)}(f_{\varepsilon_k}) - 1) \partial_x f_{\varepsilon_k}$, and since $\partial_x f_{\varepsilon_k} \rightarrow \partial_x f$ in $L_2(Q_T)$, there exists a function $F \in L_2(Q_T)$ such that, after

possibly extracting a further subsequence, $|\partial_x f_{\varepsilon_k}| \leq F$ almost everywhere in Q_T (see the proof of Theorem 3.11 in [14]). We show now that $(\chi_{(0,\infty)}(f_{\varepsilon_k}) - 1) \partial_x f_{\varepsilon_k} \rightarrow 0$ almost everywhere in Q_T . Indeed, since $\partial_x f_{\varepsilon_k} \rightarrow \partial_x f$ in $L_2(Q_T)$, we deduce that $\partial_x f_{\varepsilon_k} \rightarrow 0$ almost everywhere on the set $[f = 0]$. Furthermore, on the set $[f > 0]$, relation (3.6) implies pointwise convergence $(\chi_{(0,\infty)}(f_{\varepsilon_k}) - 1) \rightarrow 0$. Lebesgue's dominated convergence theorem implies now the desired relation (3.16), and implicitly (3.15).

To conclude, we sum (3.6), (3.14), (3.15) and let $k \rightarrow \infty$ in relation (3.13) to obtain the first identity of Theorem 1.1 (3). The corresponding relation for g follows similarly.

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